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ON ERGODIC QUASI-INVARIANT MEASURES OF GROUP AUTOMORPHISM

BY

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ABSTRACT

We study the dynamics of projective transformations and apply it to (i) prove that the isotropy subgroups of probability measures on algebraic homogeneous spaces are algebraic and to (ii) study the class of ergodic quasi-invariant measures of automorphisms of non-compact Lie groups. It is shown that their support is always a proper subset and that under certain conditions on the Lie group the induced homeomorphism of the support is topologically equivalent to a translation of a compact group.

In [4], using a simple result on the dynamics of projective transformations the author obtained a proof of Borel's density theorem in a general form. It turns out that a similar approach is useful in various other problems. It is the purpose of this article to give two more applications of the general idea.

The first application consists of strengthening a result of C. C. Moore (theorem 2.7, [9]). We assert that if G is an $\mathbf{R}$-algebraic group and H is an $\mathbf{R}$-algebraic subgroup of G then the isotropy subgroup of a finite measure μ on G/H contains a co-compact normal subgroup which fixes the support of μ pointwise (cf. Corollary 2.6). In particular, the subgroups are $\mathbf{R}$-algebraic. Further, one also obtains a criterion for amenability of the isotropy subgroup of μ in terms of the isotropy subgroups of the points in the support of μ.

The second result pertains to non-wandering sets and quasi-invariant measures of group automorphisms. We prove that if G is either isomorphic to a real vector group or if it is a connected Lie group with trivial center, A is a continuous automorphism of G, then the restriction of the action of A to the non-wandering set of A, extends to a continuous action of a compact group (cf. Theorems 3.1 and 3.2). This signifies that the ergodic dynamical systems arising from choosing various A-quasi-invariant non-atomic measures on G are all metrically conjugate to translations on compact abelian groups (cf. Corollary

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3.5). We then show that if G be any connected locally compact non-compact group then there exists a non-compact factor group G/N, by a characteristic closed subgroup N, such that G/N is a Lie group belonging to the class referred to above. Thus any continuous automorphism A of such a group has a factor automorphism having properties as in the special case (cf. Theorem 3.7) and consequently the action of A on G admits no dense orbits (Corollary 3.8); it also follows that if μ is an ergodic A-quasi-invariant measure on G then μ is supported on a proper closed subset of G (cf. Corollary 3.9).

In particular, the Haar measure of a connected locally compact non-compact group is not ergodic with respect to any continuous automorphism. This particular corollary was proved earlier by R. Kaufman and M. Rajagopalan [6] and also by R. K. Thomas [10]. The author would like to note, however, that the proof offered in [10] for the general result, Theorem 3.1, is incorrect. This is because, towards the end of the proof, it is implicitly assumed that the image of any connected Lie group under its adjoint representation is closed, which is well-known to be false. Though with some work it is possible to salvage the above-mentioned assertion of non-ergodicity of the Haar measure, indeed even with respect to any affine transformation as considered there, the "property D" does not seem to be easily recoverable. For automorphisms the "property D" evidently follows from our results (cf. Theorem 3.7 and Corollary 3.4).

Before concluding the introduction the author would like to thank H. Furstenberg for valuable suggestions; in particular, he pointed out that the results, originally formulated for quasi-invariant measures, may also be viewed fruitfully to be results about the non-wandering sets of the automorphisms.

§1. Preliminaries

Let X be a second countable topological space and φ be a homeomorphism of X onto itself. An element $x \in X$ is said to be *non-wandering* with respect to φ (or a non-wandering point of φ) if for any neighbourhood U of x there exists $n \geqslant 1$ such that $\varphi^n U \cap U$ is non-empty. The set of all non-wandering points of φ is a closed set; for brevity, it is called the *non-wandering set* of φ. We say that φ is a *non-wandering* homeomorphism of X if every element of X is non-wandering with respect to φ.

By a measure on X we shall always mean a positive σ-finite Borel measure. The *support* of a measure μ, denoted by supp μ, is the smallest closed subset of X whose complement has zero μ-measure. A measure μ on X is said to be *φ-invariant* if $\mu(\varphi(E)) = \mu(E)$ for all Borel sets; μ is said to be *φ-quasi-invariant* if for a Borel set E, $\mu(\varphi(E)) = 0$ holds if and only if $\mu(E) = 0$. A φ-

quasi-invariant measure is said to be *ergodic* if $\mu(\varphi(E)\triangle E)=0$ implies that either $\mu(E)=0$ or $\mu(X-E)=0$.

1.1. REMARKS. If μ is a φ-quasi-invariant measure then $\operatorname{supp}\mu$ is φ-invariant. If μ is a finite φ-invariant measure, then $\operatorname{supp}\mu$ is contained in the non-wandering set of φ. If μ is an ergodic φ-quasi-invariant measure then either $\operatorname{supp}\mu$ is contained in the non-wandering set of φ or there exists $x \in X$ such that the orbit $E=\{\varphi^j(x)\mid j\in\mathbf{Z}\}$ of x under φ is relatively discrete (i.e. open in its closure) and $\mu(X-E)=0$.

We shall now recall a result from [4] on the non-wandering sets of linear and projective transformations. Let V be a finite dimensional $\mathbf{R}$-vector space where $\mathbf{R}$ is the field of real numbers. We denote by $P(V)$ the projective space associated to V and by $\pi: V-(0)\to P(V)$ the natural quotient map. We denote by $\mathrm{GL}(V)$ the (topological) group of all non-singular linear transformations of V. Any $\tau\in\mathrm{GL}(V)$ induces a homeomorphism $\bar\tau$ of $P(V)$. All homeomorphisms of $P(V)$ of the form $\bar\tau$ as above are called *projective transformations*.

An element $\tau\in\mathrm{GL}(V)$ is said to be *bounded* if $\{\tau^j\mid j\in\mathbf{Z}\}$ is a relatively compact subset of $\mathrm{GL}(V)$. It is well-known that $\tau\in\mathrm{GL}(V)$ is bounded if and only if it is semisimple and all its (complex) eigenvalues are of unit absolute value. An element $\tau\in\mathrm{GL}(V)$ is said to be *projectively bounded* if there exists $\lambda\in\mathbf{R}-(0)$ such that $\lambda\tau$ is bounded. In the sense of [4], where these notions were introduced for linear transformations of (finite dimensional) vector spaces over any locally compact non-discrete field, $\tau\in\mathrm{GL}(V)$ is projectively bounded if there exists $\lambda\in\mathbf{R}-(0)$ and a positive integer m such that $\lambda\tau^m$ is bounded. It may be noted, however, that for the field of real numbers the two notions are equivalent; this may be proved easily using the observation preceding the above definition.

The result from [4] sought after is the following:

1.2. PROPOSITION (cf. lemma 4.1, [4]). *Let V be a finite dimensional $\mathbf{R}$-vector space and let $\tau\in\mathrm{GL}(V)$. Let Ω be the non-wandering set of the projective transformation $\bar\tau$. Then there exist finitely many τ-invariant subspaces $W_1, W_2,\dots, W_r$ of V such that* (i) $\Omega=\bigcup_{j=1}^r \pi(W_j-(0))$ *and* (ii) τ/W_j *is projectively bounded for all $j=1,2,\cdots,r$.*

1.3. COROLLARY. *Let τ be a linear transformation of a vector space V as above. Let W be the non-wandering set of τ. Then W is a τ-invariant subspace and τ/W is bounded. In particular, orbits of all elements in* $\operatorname{supp}\mu$ *are bounded, if μ is a finite τ-invariant measure.*

PROOF. Let Ω be the non-wandering set of $\bar{\tau}$. Then by Proposition 1.2 there exist τ-invariant subspaces $W_1, W_2, \cdots, W_r$ such that for all $j = 1, 2, \cdots, r$, τ / W_j is projectively bounded and $\Omega = \bigcup \pi(W_j - (0))$. Since, evidently, $\pi(W - (0))$ is contained in Ω we now deduce that W is contained in $\bigcup W_j$. Fix an index j between 1 and r such that $W \cap W_j \neq 0$. We recall that since τ / W_j is projectively bounded it has the form $\lambda_j \sigma_j$ where $\lambda_j \in \mathbf{R} - (0)$ and σ_j is a bounded linear transformation of W_j. On W_j there exists a norm which is invariant under σ_j. Using this norm it is straightforward to check that the transformation $\lambda_j \sigma_j$ admits a non-wandering point in W_j other than zero, only if $|\lambda_j| = 1$. We deduce that τ / W_j is bounded. Now let W' be the subspace spanned by $\{W_j \mid W \cap W_j \neq 0\}$. Then clearly τ / W' is bounded and W is contained in W'. On the other hand, since for a bounded linear transformation every point is non-wandering, W' is contained in W. Thus $W = W'$, which yields the corollary.

§2. Group actions and measures

We shall now consider actions of groups of linear and projective transformations. Let V be a finite dimensional $\mathbf{R}$-vector space and let $P(V)$, $\mathrm{GL}(V)$ etc. be as in §1. $\mathrm{GL}(V)$ acts as a group of linear transformations on V and as a group of projective transformations on $P(V)$. In the sequel we consider actions on V and $P(V)$ simultaneously and by the action of $\mathrm{GL}(V)$, or any of its subgroups, we mean the respective actions as above. Let F be a non-empty closed subset, either of V or of $P(V)$. Let L be the smallest quasi-linear variety (i.e. finite union of subspaces) such that F is contained in L, or, respectively, in $\pi(L - (0))$. For $g \in \mathrm{GL}(V)$ such that $g(F) = F$, we denote by g/F the restriction of the g-action to F. Now put

$$C_F = \{g \in \mathrm{GL}(V) \mid g(F) = F \text{ and } g/F \text{ is a non-wandering homeomorphism}\},$$

$$I_F = \{g \in \mathrm{GL}(V) \mid gx = x \text{ for all } x \in F\}, \tag{2.1}$$

$$N_F = \{g \in \mathrm{GL}(V) \mid g(L) = L\}.$$

Then I_F and N_F are $\mathbf{R}$-algebraic subgroups of $\mathrm{GL}(V)$ (i.e. intersections with $\mathrm{GL}(V)$ of algebraic $\mathbf{R}$-subgroups of $\mathrm{GL}(V \otimes \mathbf{C})$). Further I_F is a normal subgroup of N_F. In general C_F may not be a subgroup. However, $g \in C_F$ implies that $g^r \in C_F$ for all $r \in \mathbf{Z}$. We also note that $I_F \subset C_F \subset N_F$.

2.2. THEOREM. *Every element of C_F / I_F is contained in a compact subgroup of N_F / I_F. If H is a closed subgroup such that $I_F \subset H \subset C_F$ then H / I_F is compact.*

PROOF. Let $W_1, W_2, \cdots, W_r$ be the maximal subspaces of L. Let N_F^0 be the subgroup of those elements of N_F which leave each W_j invariant. We note that N_F^0 is a subgroup of finite index in N_F. Consider the homomorphism $\varrho : N_F^0/I_F \to \mathrm{GL}(W_1)\times \mathrm{GL}(W_2)\times\cdots\times \mathrm{GL}(W_r)$ where for $g \in N_F^0$ the jth component of $\varrho(gI_F)$ is defined to be the restriction of g to W_j. Since N_F^0 is an $\mathbf{R}$-algebraic subgroup, the image of ϱ is a closed subgroup of the product (cf. Lemma 2.3 below). We now show that for all $g \in C_F \cap N_F^0$, gI_F is contained in a compact subgroup of N_F^0/I_F. Consider first the case when F is a (closed) subset of V. In this case, by Corollary 1.3, the components of $\varrho(gI_F)$ are bounded. Hence $\varrho(gI_F)$ is contained in a compact subgroup of the image of ϱ. Also, in the present case the kernel of ϱ is precisely I_F. Hence we deduce that gI_F is contained in a compact subgroup of N_F^0/I_F. Next suppose that F is a subset of $P(V)$. In this case by Proposition 1.2 for $g \in C_F \cap N_F^0$, $\varrho(gI_F)$ is of the form $(\lambda_1\sigma_1, \lambda_2\sigma_2, \cdots, \lambda_r\sigma_r)$ where $\lambda_1, \lambda_2, \cdots, \lambda_r$ are positive scalars and $\sigma_1, \sigma_2, \cdots, \sigma_r$ are bounded. It is straightforward to show that $(\lambda_1 I_1, \lambda_2 I_2, \cdots, \lambda_r I_r)$, where $I_1, I_2, \cdots, I_r$ denote the identity transformations of the respective spaces, is contained in the image of ϱ. Hence $(\sigma_1, \sigma_2, \cdots, \sigma_r)$ is also contained in the image of ϱ. Further, it must indeed be contained in a compact subgroup of the image. Since $\varrho^{-1}(\lambda_1 I_1, \lambda_2 I_2, \cdots, \lambda_r I_r)$ as well as the kernel of ϱ are contained in I_F we deduce that gI_F is contained in a compact subgroup of N_F^0/I_F. Since N_F^0 has finite index in N_F it now follows that for all $g \in C_F$, gI_F is contained in a compact subgroup of N_F/I_F.

Now let H be a closed subgroup as in the hypothesis. Then in view of the above every element of H/I_F is contained in a compact subgroup of N_F/I_F. But N_F/I_F is a closed subgroup of an algebraic group (cf. theorem 5.6 in [1] and Lemma 2.3 below). Hence, in particular, H/I_F may be viewed as a closed subgroup of $\mathrm{GL}(n, \mathbf{R})$ for some n. The compactness of H/I_F now follows from a lemma of C. C. Moore (cf. lemma 7.1 in [8]).

In the sequel we need an analogue of the above theorem for a wider class of groups, in the place of $\mathrm{GL}(V)$. A subgroup G of $\mathrm{GL}(V)$ is said to be *almost algebraic* if G is open in an $\mathbf{R}$-algebraic subgroup of $\mathrm{GL}(V)$. We note that in this case G has finite index in its Zariski closure in $\mathrm{GL}(V)$. The following lemma is well-known; we include a brief proof for ready reference.

2.3. LEMMA. *Let G be the group of $\mathbf{R}$-elements of an algebraic group $\tilde{G}$ defined over $\mathbf{R}$. Let $\varrho : \tilde{G} \to \mathrm{GL}(V_{\mathbf{C}})$, where $V_{\mathbf{C}} = V \otimes \mathbf{C}$, be an (algebraic) representation defined over $\mathbf{R}$. Then $\varrho(G)$ is an almost algebraic subgroup of* $\mathrm{GL}(V)$. *In particular, it is a closed subgroup of* $\mathrm{GL}(V)$.

PROOF. It is well-known that $\varrho(\tilde{G})$ is an algebraic subgroup of $\mathrm{GL}(V_{\mathbf{C}})$

defined over **R** (cf. corollary 1.4, [1]). From this, one deduces that the **C**-dimension of $\varrho(\tilde{G})$ coincides with the dimension of $\varrho(G)$ as a Lie group. Hence by the inverse function theorem $\varrho(G)$ is an open subgroup of $\varrho(\tilde{G}) \cap \mathrm{GL}(V)$. Since $\varrho(\tilde{G})$ is an algebraic group it now follows that $\varrho(G)$ is an almost algebraic subgroup of $\mathrm{GL}(V)$.

2.4. Corollary. *Let V be a finite dimensional* **R**-*vector space. Let F be a closed subset of either V or $P(V)$ (notations as before). Let G be an almost algebraic subgroup of* $\mathrm{GL}(V)$. *Then every element of $(C_F \cap G)/(I_F \cap G)$ is contained in a compact subgroup of $(N_F \cap G)/(I_F \cap G)$.*

Proof. Let $\tau \in C_F \cap G$. Let H be the closed subgroup of $\mathrm{GL}(V)$ generated by τ and I_F. Since $\tau \in C_F$, by Theorem 2.2 H/I_F is compact. Using the fact that a compact subgroup of a real linear group is an **R**-algebraic subgroup (cf. chapter VI, [3]), one deduces that H itself is an **R**-algebraic subgroup of $\mathrm{GL}(V)$. Hence $H \cap G$ is an almost algebraic subgroup of $\mathrm{GL}(V)$. Therefore $(H \cap G)I_F$ is a closed subgroup. Since H/I_F is compact so is $(H \cap G)I_F/I_F$. In other words $(H \cap G)/(I_F \cap G)$ is compact, which proves the corollary.

2.5. Corollary. *Let V be a finite dimensional* **R**-*vector space and let μ be a finite measure on V or $P(V)$. Let G be an almost algebraic subgroup of* $\mathrm{GL}(V)$. *Let G_μ be the subgroup consisting of all elements of G whose action leaves μ invariant. Let $I_\mu = I_F$, where $F =$ support of μ. Then $G_\mu / G_\mu \cap I_\mu$ is compact.*

Proof. Let $\mathrm{GL}(V)_\mu$ be the subgroup consisting of all $g \in \mathrm{GL}(V)$ such that μ is invariant under the action of g. Then $\mathrm{GL}(V)_\mu$ is a closed subgroup containing I_μ. Further, by Remark 1.1 $\mathrm{GL}(V)_\mu$ is contained in C_F as defined by (2.1), where $F =$ support of μ. Hence by Theorem 2.2, $\mathrm{GL}(V)_\mu / I_\mu$ is compact. As in the proof of Corollary 2.4 this implies that $\mathrm{GL}(V)_\mu$ is an **R**-algebraic subgroup. Hence $G_\mu = G \cap \mathrm{GL}(V)_\mu$ is an almost algebraic subgroup. Hence $G_\mu I_\mu$ is a closed subgroup and $G_\mu I_\mu / I_\mu$ is compact. Therefore $G_\mu / G_\mu \cap I_\mu$ is compact.

The following corollary generalises a result of C. C. Moore (cf. Theorem 2.7, [9]).

2.6. Corollary. *Let G and H be almost algebraic subgroups of* $\mathrm{GL}(n, \mathbf{R})$, *for some $n \geqq 1$, and suppose that H is contained in G. Let μ be a finite measure on G/H. Let*

$$G_\mu = \{g \in G \mid \textit{the } g\textit{-action on } G/H \textit{ preserves } \mu\}$$

and

$$J_\mu = \{g \in G \mid gx = x \textit{ for all } x \in \operatorname{supp} \mu\}$$

(where the action refers to the action by translation on the left). Then G_μ is an **R***-algebraic subgroup of G containing J_μ as a normal subgroup and G_μ / J_μ is compact. In particular, G_μ is an amenable group if and only if J_μ is amenable.*

PROOF. Under the above hypothesis there exists a representation ϱ of G over a finite dimensional **R**-vector space V and a non-zero vector $v \in V$ such that H is the isotropy subgroup of either $v \in V$ or $\pi(v) \in P(V)$, under the respective actions induced by ϱ (cf. for instance, proposition 7.8, [2]). The measure μ on G/H can be canonically identified with a measure on the G-orbit of v or $\pi(v)$ respectively. Application of Corollary 2.5 then yields that G_μ / J_μ is compact. Since J_μ is an **R**-algebraic subgroup, as in the proof of Corollary 2.4 we deduce that G_μ is also **R**-algebraic. The assertion about amenability is obvious.

§3. Quasi-invariant measures of group automorphisms

We first study quasi-invariant measures of automorphisms of certain special groups and combine it to obtain certain general results. We note that by an automorphism we always mean a continuous automorphism.

3.1. THEOREM. *Let G be a group topologically isomorphic to $\mathbf{R}^n$, $n \geqq 1$. Let A be an automorphism of G and let F be the non-wandering set of A. Then there exists a continuous action of a compact group K on F and $k \in K$ whose action coincides with that of A.*

PROOF. In this case G has the structure of an n-dimensional **R**-vector space such that the automorphism A is a linear transformation. By Corollary 1.3, F is an A-invariant subspace and the restriction A/F of A to F is a bounded linear transformation. Let K be the closed subgroup of GL(F) generated by A/F. Then K is a compact group with a continuous action on F, extending that of A.

3.2. THEOREM. *Let G be a connected Lie group with trivial center, A an automorphism of G and let F be the non-wandering set of A. Then there exist a continuous action of a compact group K on F and $k \in K$ whose action coincides with that of A.*

PROOF. Let $\tilde{G}$ be the universal covering group of G. Let Z denote the center of $\tilde{G}$. Since G has trivial center it follows that Z is discrete and that G is topologically isomorphic to $\tilde{G}/Z$. We shall identify G with $\tilde{G}/Z$.

Let Aut($\tilde{G}$) be the (topological) group of all Lie automorphisms of $\tilde{G}$. Since $\tilde{G}$ is simply connected, Aut($\tilde{G}$) is isomorphic to the group of all Lie automorphisms of the Lie algebra $\mathfrak{G}$ of G. In particular Aut($\tilde{G}$) is an **R**-algebraic subgroup of GL($\mathfrak{G}$). Let Aff($\tilde{G}$) be the group of all affine automorphisms $T_x \cdot \tau$ where

$\tau \in \mathrm{Aut}(\tilde{G})$ and T_x denotes the left translation of $\tilde{G}$ by $x \in \tilde{G}$. We shall identify $\tilde{G}$ with the subgroup of $\mathrm{Aff}(\tilde{G})$ consisting of all left translations. $\mathrm{Aff}(\tilde{G})$ is then a semi-direct product of $\mathrm{Aut}(\tilde{G})$ and $\tilde{G}$. With respect to the product analytic structure $\mathrm{Aff}(\tilde{G})$ is a (not necessarily connected) Lie group.

We now construct a representation ϱ of $\mathrm{Aff}(\tilde{G})$ as follows. Let $\mathrm{Aff}(\mathfrak{G})$ denote the Lie algebra of $\mathrm{Aff}(\tilde{G})$ and let $\mathrm{Aut}(\mathfrak{G})$ be the Lie subalgebra corresponding to the subgroup $\mathrm{Aut}(\tilde{G})$. Let q be the dimension of $\mathrm{Aut}(\mathfrak{G})$. Evidently, $q \geqq 1$. Set $V = \Lambda^q \mathrm{Aff}(\mathfrak{G})$, the qth exterior power of $\mathrm{Aff}(\mathfrak{G})$ (as a vector space). Let $\varrho : \mathrm{Aff}(\tilde{G}) \rightarrow GL(V)$ be the qth exterior power of the adjoint representation of $\mathrm{Aff}(\tilde{G})$. Let u be a vector in V generating the 1-dimensional subspace $\Lambda^q \mathrm{Aut}\, \mathfrak{G}$. We note here that the restriction of ϱ to $\mathrm{Aut}(\tilde{G})$ is the restriction of an algebraic $\mathbf{R}$-representation of the respective Zariski closures; this can be seen as follows: The adjoint representation of $\mathrm{Aut}(\tilde{G})$ on $\mathrm{Aut}(\mathfrak{G})$ is the restriction of an $\mathbf{R}$-representation (cf. pp. 127, [1]) and so is the natural representation of $\mathrm{Aut}(\tilde{G})$ on $\mathfrak{G}$. Since the restriction of ϱ to $\mathrm{Aut}(\tilde{G})$ is an exterior power of the direct sum of these representations, it is the restriction of an $\mathbf{R}$-representation.

Let $P(V)$ be the projective space associated to V. Let $\pi : V-(0) \rightarrow P(V)$ be the quotient map and let $\theta = \pi(u)$. Consider the action of $\mathrm{Aff}(\tilde{G})$ on $P(V)$ induced by ϱ. For $\xi \in \mathrm{Aff}(\tilde{G})$ we shall denote by $\varphi(\xi)$ the homeomorphism of $P(V)$ induced by ξ; i.e. $\varphi(\xi)\pi(v) = \pi(\rho(\xi)v)$ for all $v \in V-(0)$. Now consider the isotropy subgroup, say J, of θ under the action of $\mathrm{Aff}(\tilde{G})$. It is straightforward to verify that if $\xi \in \mathrm{Aff}(\tilde{G})$ then $\xi \in J$ if and only if $\mathrm{Aut}(\mathfrak{G})$ is invariant under the adjoint action of ξ. Thus, in particular, $\mathrm{Aut}(\tilde{G})$ is contained in J. Now let $\xi \in \tilde{G} \cap J$. Then in view of the above, ξ must normalise $\mathrm{Aut}^0(\tilde{G})$, the connected component of the identity in $\mathrm{Aut}(\tilde{G})$. Let $\tau \in \mathrm{Aut}^0(\tilde{G})$ and consider $\sigma = \tau \cdot \xi \cdot \tau^{-1} \cdot \xi^{-1} \in \mathrm{Aff}(\tilde{G})$. Since ξ normalises $\mathrm{Aut}^0(\tilde{G})$, $\sigma \in \mathrm{Aut}^0(\tilde{G})$. On the other hand, since $\tilde{G}$ is normal in $\mathrm{Aff}(\tilde{G})$, $\sigma \in \tilde{G}$. Hence σ is the identity transformation. Thus for all $\tau \in \mathrm{Aut}^0(\tilde{G})$ we have $\tau \cdot \xi \cdot \tau^{-1} = \xi$. But clearly $\tau \cdot \xi \cdot \tau^{-1} = \tau(\xi)$, as an element of $\tilde{G}$. Since the group of inner automorphisms of G is contained in $\mathrm{Aut}^0(\tilde{G})$ we deduce that ξ as above must be contained in the center of $\tilde{G}$. Thus $\tilde{G} \cap J$ is contained in Z.

We next prove that Z is contained in J. Z being finitely generated the subgroup of $\mathrm{Aut}^0(\tilde{G})$ consisting of all those automorphisms whose restriction to Z is the identity transformation is a subgroup of countable index in $\mathrm{Aut}^0(\tilde{G})$. Since it is a closed subgroup and $\mathrm{Aut}^0(\tilde{G})$ is connected the last assertion implies that the restriction of each $\tau \in \mathrm{Aut}^0(\tilde{G})$ to Z is the identity transformation. Thus for any $z \in Z$ and $\tau \in \mathrm{Aut}^0(\tilde{G})$ we have $\tau \cdot z \cdot \tau^{-1} = \tau(z) = z$. In particular, z normalises $\mathrm{Aut}^0(\tilde{G})$. Hence by our earlier remark Z is contained in J.

Combining the results, we deduce that $J = \mathrm{Aut}(\tilde{G}) \cdot Z$. Now let $\alpha : G \to P(V)$ be the map defined by $\alpha(g) = \varphi(\tilde{g})\theta$ where $\tilde{g}$ is a representative for $g \in G = \tilde{G}/Z$ in $\tilde{G}$. In view of the above, α is a well-defined continuous one-one map. Let $\tau \in \mathrm{Aut}(\tilde{G})$ and let $\bar{\tau}$ be the factor automorphism of τ on G. Let $g = xZ/Z \in G$. Then we have

$$\begin{aligned}\varphi(\tau)\alpha(g) = \varphi(\tau)\varphi(x)\theta &= \varphi(\tau \cdot x)\theta = \varphi(\tau \cdot x \cdot \tau^{-1})\varphi(\tau)\theta \\ &= \varphi(\tau \cdot x \cdot \tau^{-1})\theta = \varphi(\tau(x))\theta = \alpha(\bar{\tau}(g)).\end{aligned}$$

In other words for all $\tau \in \mathrm{Aut}(\tilde{G})$ we have

$$\varphi(\tau) \cdot \alpha = \alpha \cdot \bar{\tau} \tag{3.3}$$

where $\bar{\tau}$ is the factor automorphism of τ on G.

Now let A be an automorphism of G. Let $\tilde{A}$ be the unique automorphism of $\tilde{G}$ which has A as its factor on G. Let F be the non-wandering set of A. Let E be the closure of $\alpha(F)$ in $P(V)$. Using (3.3) it is easy to verify that E is $\varphi(\tilde{A})$-invariant and that the restriction of $\varphi(\tilde{A})$ to E is non-wandering. Now let I_E and N_E be the subgroups of $\mathrm{GL}(V)$ defined by (2.1), for E in the place of F. Let $\mathcal{G} = \varrho(\mathrm{Aut}(\tilde{G}))$. Since the restriction of ϱ to $\mathrm{Aut}(\tilde{G})$ coincides with the restriction of an (algebraic) **R**-representation, by Lemma 2.3 $\mathcal{G}$ is an almost algebraic subgroup of $\mathrm{GL}(V)$. Therefore by Corollary 2.4, if H is the closed subgroup generated by $\varrho(\tilde{A})$ and $I_E \cap \mathcal{G}$, then $I_E \cap \mathcal{G}$ is normal in H and $H/I_E \cap \mathcal{G}$ is a compact group. Let $I'_E = \varrho^{-1}(I_E \cap \mathcal{G})$ and let H' be the closed subgroup of $\mathrm{Aut}(\tilde{G})$ generated by $\tilde{A}$ and I'_E. Then in view of the above, I'_E is normal in H' and H'/I'_E is a compact group.

We shall now obtain a continuous action of H'/I'_E on F extending the action of A. For any $\tau \in H'$ let $\bar{\tau}$ denote the factor automorphism of τ on G. Let $\sigma \in I'_E$ be arbitrary. Then $\varphi(\sigma)$ fixes every point in E. Recall that α is a one-one map and that $\alpha(F)$ is contained in E. In view of (3.3), the previous assertion therefore implies that $\bar{\sigma}$ fixes every point in F. Also F is invariant under A, which indeed is the factor automorphism of $\tilde{A}$. Since F is a closed set, from the last two assertions in particular it may be deduced that for all $\tau \in H'$ and $y \in F$, $\tau(y)$ is contained in F. We therefore get an action of the compact group $K = H'/I'_E$ defined by $(k, y) \mapsto \bar{\tau}(y)$ for all $y \in F$ and $k \in K$, $\tau \in H'$ being any representative of k. The action is clearly well defined and continuous. Further, the homeomorphism corresponding to $\tilde{A}I'_E/I'_E$ evidently coincides with the action of A.

3.4. COROLLARY. *Let G be either* $\mathbf{R}^n$ *or a connected Lie group with trivial* *center. Let A be an automorphism of G and let F be the non-wandering set of A. Then any compact subset of F is contained in a compact A-invariant subset. Further, the action of $\{A^j \mid j \in \mathbf{Z}\}$ on any compact A-invariant set is equicontinuous.*

PROOF. Is obvious from Theorems 3.1 and 3.2.

3.5. COROLLARY. *Let G and A be as in the above corollary. Let μ be a measure on G which is quasi-invariant and ergodic under the action of A. Suppose that there does not exist a relatively discrete orbit E such that $\mu(E) > 0$. Then we have the following:*

(i) *The support of μ is compact.*

(ii) *Let X = support of μ. Then X is A-invariant and the restriction of A to X is topologically conjugate to a translation on a compact abelian group; i.e. there exist a compact abelian group T, $a \in T$ and a homeomorphism φ of T onto X such that $\varphi(at) = A(\varphi(t))$ for all $t \in T$.*

(iii) *If, further, μ is A-invariant and finite then under (any) φ as in* (ii) *μ is the image of the Haar measure m on T, with appropriate total measure; i.e. φ induces a metrical isomorphism of the translation by a of (T, m) and the automorphism A of (G, μ).*

PROOF. Under the hypothesis on μ, as above, by Remark 1.1 the support of μ is A-invariant and is contained in the non-wandering set of A. Hence by Theorems 3.1 and 3.2 there exists a continuous action of a compact abelian group K on supp μ extending the action of A. The partition of supp μ into orbits of K is a countably separated partition (cf. [5] for a more general result). Since μ is ergodic with respect to the action of A we get that μ is concentrated on a single orbit of K. Thus supp μ is a single orbit of K. Hence assertion (i) is obvious. Next let $x_0 \in X = \operatorname{supp} \mu$ and let I be the isotropy subgroup of x_0 under the K action. Put $T = K/I$. Let k be the element of K whose action on X coincides with that of A, and let $a \in T$ be the coset of k. Let $\varphi : T \to X$ be the homeomorphism defined by $\varphi(t) = \tilde{t}x_0$ where $t \in T$ and $\tilde{t}$ is any representative of t in K. Then evidently assertion (ii) is satisfied. Assertion (iii) follows easily from the fact that (up to a scalar multiple) the Haar measure of T is the only finite measure invariant under translation by a.

We shall next deduce certain consequences of the above corollaries for automorphisms of more general groups including all Lie groups. We need the following.

3.6. LEMMA. *Let G be a connected non-compact Lie group. Then there exists a* *closed characteristic subgroup N of G such that G/N is either* (a) *topologically isomorphic to* $\mathbf{R}^n$, *with* $n \geqq 1$, *or* (b) *a non-compact Lie group with trivial center.*

PROOF. We define inductively a sequence $\{G_i\}_0^\infty$ of quotients of G as follows: Set $G_0 = G$ and for $i \in \mathbf{N}$ let $G_i = G_{i-1}/Z_{i-1}$ where Z_{i-1} is the center of G_{i-1}. Then each G_i is a Lie group of the form G/N_i where N_i is a closed characteristic subgroup of G. Let $m \in \mathbf{N}$ be such that G_m has least possible dimension. Then clearly for all $r \geqq m$, Z_r is a discrete subgroup of G_r. Let $\eta : G_m/Z_m \to G_{m+1}$ be the natural quotient map. Then $\eta^{-1}(Z_{m+1})$ is a discrete normal subgroup of G_m. Since the latter is connected it follows that $\eta^{-1}(Z_{m+1})$ is contained in its center. In other words, $\eta^{-1}(Z_{m+1})$ is contained in Z_m and hence Z_{m+1} is trivial.

If G_{m+1} is non-compact then in view of the above we are through. Now suppose that G_{m+1} is compact. Let $0 \leqq r \leqq m$ be the smallest integer such that G_{r+1} is compact. As a connected Lie group whose quotient by its center is compact, G_r is isomorphic to $\mathbf{R}^n \times C$ (direct product) where $n \geqq 0$ and C is a compact group. Since by choice G_r is non-compact $n \geqq 1$. It is now evident that G admits a characteristic subgroup N such that G/N is topologically isomorphic to $\mathbf{R}^n$, where $n \geqq 1$.

3.7. THEOREM. *Let G be a connected, locally compact, non-compact topological group. Then there exists a closed characteristic subgroup N of G such that G/N is either* (*topologically isomorphic to*) $\mathbf{R}^n$, $n \geqq 1$ *or a non-compact Lie group with trivial center, and the following assertion holds*: *Let A be a continuous automorphism of G and let F be the non-wandering set of A. Let Y be the closure of the image of F in G/N under the quotient map. Then there exists a continuous action of a compact group K on Y extending the action of the factor automorphism. Also, if μ is an A-quasi-invariant ergodic measure on G such that there exists no relatively discrete orbit of positive μ-measure, then the image of* supp μ *in G/N is a bounded subset.*

PROOF. Under the above hypothesis on G there exists a unique maximum compact normal subgroup M of G such that G/M is a Lie group (cf. chapter IV, [7]). It is easy to prove that M is a characteristic subgroup of G (i.e. $A(M) = M$ for any continuous automorphism A of G). Combining this with Lemma 3.6 we deduce that there exists a closed characteristic subgroup N of G such that G/N is either $\mathbf{R}^n$, $n \geqq 1$ or a non-compact Lie group with trivial center. The theorem now follows from Theorems 3.1 and 3.2 and Corollary 3.5; it is enough to note that Y is contained in the non-wandering set of the factor automorphism of A on G/N.

3.8. Corollary. *Let G be a locally connected locally compact non-compact group and A be a (continuous) automorphism of G. Then the action of A on G admits no dense orbit; i.e. $\{A^j(x) \mid j \in \mathbf{Z}\}$ is not dense in G for any $x \in G$.*

Proof. If possible let $x \in G$ be such that $E = \{A^j(x) \mid j \in \mathbf{Z}\}$ is dense in G. Since G is locally connected, the connected component G^0 of the identity in G is an open subgroup of G. Hence there exists $j \in \mathbf{Z}$ such that $A^j(x) \in G^0$. Since G^0 is A-invariant we deduce that $x \in G^0$ and that $G^0 = G$; i.e. G is connected. Now by passing to quotient by N as in Theorem 3.7 we may assume that G is a non-compact group satisfying the hypothesis of Corollary 3.4. Suppose $x \in G$ is such that its orbit under A is dense. Then x is a non-wandering point for A. By Corollary 3.4 the orbit of x must be bounded. This implies that G is compact, contradicting the hypothesis. Hence the corollary.

3.9. Corollary. *Let G be a locally connected, locally compact non-compact topological group and let A be a continuous automorphism of G. If μ is an A-quasi-invariant ergodic (σ-finite) measure then it is supported on a proper subset of G; i.e. there exists a non-empty open subset Ω of G such that $\mu(\Omega) = 0$. In particular, the Haar measure of G is not ergodic with respect to any continuous automorphism of G.*

Proof. This can be deduced from Corollary 3.8 using standard techniques. We omit the details.

§4. Comments and questions

It seems reasonable to the author to expect that any non-atomic ergodic quasi-invariant measure of an automorphism of any connected Lie group would have compact support. Further, the analogues of assertions (ii) and (iii) in Corollary 3.5 may be expected to hold for any Lie group with no non-trivial compact normal subgroup.

As mentioned in the introduction, Corollary 3.9 is also true for affine automorphisms in the place of automorphisms. The author has recently obtained a direct proof of the analogue of Corollary 3.8 for all affine automorphisms of connected locally compact groups, which will appear elsewhere.

At present the author has certain technical difficulties in proving the analogue of Theorem 3.2 for affine transformations. We note, however, that by imitating the proof of Theorem 3.2 it is easy to prove the following.

4.1. Theorem. *Let G be an almost algebraic subgroup of* $\mathrm{GL}(n, \mathbf{R})$, $n \geqq 1$.

Suppose that the group of all automorphisms of G is an almost algebraic subgroup (when viewed as a group of linear automorphisms of the Lie algebra of G). Then any non-atomic measure on G which is quasi-invariant and ergodic under the action of an affine transformation of G, has compact support.

Added in proof. The analogue of Corollary 3.8 referred to above may be found in J. Lond. Math. Soc. **25** (1982), 241–245.

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